KINETIC THEORY OF SHOCK-STRUCTURE USING SMALL PERTURBATION TECHNIQUE

by

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1. *Introduction.*

Considerable effort has been expended in recent years in an endeavour to understand the shock-structure problem and surveys concerning the shock-structure problem can be found in [1, 2, 3]. Much progress in this direction has been made in recent years and the account is given in references [4-9]. In general there have been two trends in this direction. Firstly, the Boltzmann equation is reduced to the equations of macroscopic variables for general gas dynamics by the method of iteration initiated by Maxwell and these equations are applied to shock-waves under special conditions. Secondly, the method consists in seeking a special function which may imply the main characteristics of the distribution within a shock-wave. The unknown variables involved in this function are determined, so that the function will satisfy equations of moments of the same number as that of the unknown variables. The method of Mott-Smith seems to be the only one which belongs to this category. It seems that there is no sequence of logic which enables one to find such a function. "The weapon seems to be a complex of knowledge accumulated by experiences, which might be called intuition".

The term "structure" as applied to gas dynamic discontinuity refers to the values of the physical properties of the fluid within the small but finite thickness of the discontinuity. The most important gas dynamic discontinuity is the shock-wave, in which there are abrupt changes in the physical conditions (viz. velocity, temperature, etc. etc.) of a moving fluid. Such abrupt changes can occur only when flow velocities exceed the acoustic velocity and hence are characteristic of supersonic flows. The actual zone within which transition from one physical state to a second takes place are finite but extremely narrow (of the order of several mean free paths). The process or mechanism that results in this transition is very complex from a physical as well as mathematical point of view. Further, the problem of shock-wave structure is not beset by the mysteries of gas-solid interactions on a molecular scale. Thus surface effects such as slip do not give rise to any sort of complications. The solution of Boltzmann equation is, in general, a matter of considerable difficulty, even in cases corresponding to the physically simplest situation.

The problems in kinetic theory of gases are complicated to handle in practise because of the intractable nature of the Boltzmann equation's binary collision term. For the general case, one has a non-linear, integro-differential equation, the integral involving a fivefold integration. Up to now various methods have been employed to study the boundary value or the initial value problems connected with Boltzmann equation and the salient features of these methods are sketched in [10]. We employ the kinetic model of Boltzmann equation, commonly known as the Krook model. The Krook model is considerably simpler than the standard Boltzmann equation, since the distribution function enters into the collision terms in a simple way. The mathematical simplification introduced with the model enables one to solve problems which are physically more complex than those soluble with the standard Boltzmann equation; in addition, one is able to treat definite initial and boundary-value problems.

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Moreover the model possesses at least the minimum requirements of a meaningful kinetic equation in spite of its obvious limitations as for instance disagreement with kinetic theory on the ratio of viscosity to heat conduction coefficients. It has the required five collisional invariants, it has the H-Theorem and it reduces to the equilibrium (max.) distribution in the long time limit of the spatially homogeneous case. More details about the model can be found in [11].

In the present work is introduced a new approximation procedure based on the use of small perturbation technique for the solution of kinetic model equation in case of shock-structure problem. The molecules of the gas are considered to be monatomic but with no internal degrees of freedom. The influence of external forces on the distribution function is neglected. The results are shown to be in good agreement with those obtained by the other methods. The method is applicable in case of weak shocks and as far as the author is aware, the method of small perturbation technique was never applied for the solution of the problem of shock-wave structure. The calculations are simpler although quite tedious and cumbersome.

2. *Basic Equation and Relations.*

The Krook model equation is written as

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} = An(F - f) \tag{1}$$

where $f(\underline{r}, \underline{v}, t)$ is the distribution function and $\underline{v}$, $\underline{r}$ and t are the molecular velocity vector, the space vector and time respectively. The number A is a free parameter which in general may depend on the state of the gas.

F in equation (1) is the locally Maxwellian distribution function represented by

$$F = n \left[\frac{m}{2k\pi T}\right]^{3/2} \exp\left[-\frac{m}{2kT}(\underline{v} - \underline{u})^2\right] \tag{2}$$

where $\underline{u}$ is the mass velocity of the flow and m, k and T are the mass of the particle, the Boltzmann constant and the absolute temperature respectively. Here n, $\underline{u}$ and T are, in general, functions of $\underline{r}$ and t, and given by

$$n = \int f d\underline{v} \tag{3}$$

$$\underline{u} = \frac{1}{n}\int \underline{v}\, f d\underline{v} \tag{4}$$

$$\frac{3}{2\beta} = \frac{3kT}{m} = \frac{1}{n}\int (\underline{v} - \underline{u})^2 f d\underline{v} \tag{5}$$

where the integrals are evaluated over the full range of the molecular velocities.

We take shock-fixed coordinates with the velocity u in the positive x-direction. (fig. 1).

The state ahead of the shock-wave, i.e. for $x \to -\infty$, will be denoted by (1), the downstream state $x \to +\infty$ by (2), viz.

$$n(-\infty) = n_1, \ u(-\infty) = u_1 \ \text{etc.}$$

and $n(+\infty) = n_2, \; u(+\infty) = u_2$ etc.

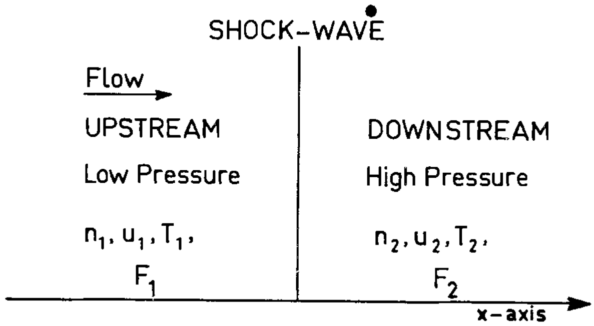

Fig. 1. Notation.

Equation (1) for plane normal shock can be written as

$$v_x \frac{df}{dx} = An(F - f) \tag{6}$$

Integrating this formally for $v_x \gtrless 0$, respectively, and imposing the boundary conditions

$$f(-\infty; \underline{v}) = F_1 = n_1 \left(\frac{\beta_1}{\pi}\right)^{3/2} \exp\left[-\beta_1(\underline{v} - \underline{u}_1)^2\right] \tag{7}$$

$$f(+\infty; \underline{v}) = F_2 = n_2 \left(\frac{\beta_2}{\pi}\right)^{3/2} \exp\left[-\beta_2(\underline{v} - \underline{u}_2)^2\right] \tag{8}$$

where the parameters are determined from the Rankine-Hugoniot conditions, we obtain expressions for the half-range distribution functions i.e.,

$$f(x; v_x > 0) = f_+ = \int_{-\infty}^{x} \frac{AnF}{v_x} \left[\exp\left(-\int_{x'}^{x} \frac{Andx''}{v_x}\right)\right] dx' \tag{9}$$

$$f(x; v_x < 0) = f_- = \int_{+\infty}^{x} \frac{AnF}{v_x} \left[\exp\left(-\int_{x'}^{x} \frac{Andx''}{v_x}\right)\right] dx' \tag{10}$$

where we have omitted the "complementary-function" solution of (6) as the boundaries are at infinity. Far away from the shock $f(\mp\infty), \underline{v}) = F_{1,2}(\underline{v})$ irrespective of the sign of v_x, and the boundary conditions will be satisfied. In other words at large distances from the shock the local flow parameters tend to the respective constant values at the boundary, the deviations from Maxwellian becomes increasingly small and at the limit the distribution function is identical to the Maxwellian and the boundary conditions are satisfied. The discussion of this point is given in [5], [7] and [12].

Inserting the values of f in (3), (4) and (5) we obtain

$$n(x) = A\left(\frac{m}{2k\pi}\right)^{3/2} \left[\int_{-\infty}^{x} \frac{n^2}{T^{3/2}} \int_{0}^{\infty} \frac{1}{v_x} \exp\left\{ -\int_{x'}^{x} \frac{An}{v_x} dx'' - \frac{m}{2kT}(\underline{v} - \underline{u})^2 \right\} + \right.$$

$$\left. + \int_{x}^{\infty} \frac{n^2}{T^{3/2}} \int_{0}^{\infty} \frac{1}{v_x} \exp\left\{ -\int_{x'}^{x} \frac{An}{v_x} dx'' - \frac{m}{2kT}(\underline{v} - \underline{u})^2 \right\} \right] d\underline{v}dx'. \tag{11}$$

$$n(x)u(x) = A\left(\frac{m}{2k\pi}\right)^{3/2}\Bigg[\int_{-\infty}^{x}\frac{n^2}{T^{3/2}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(\underline{v}-\underline{u})^2\right\} +$$

$$+\int_x^{\infty}\frac{n^2}{T^{3/2}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(\underline{v}-\underline{u})^2\right\}\Bigg]\underline{v}\,d\underline{v}\,dx'. \quad (12)$$

$$\frac{3k}{m}\cdot n(x)T(x) = A\left(\frac{m}{2k\pi}\right)^{3/2}\Bigg[\int_{-\infty}^{x}\frac{n^2}{T^{3/2}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(\underline{v}-\underline{u})^2\right\}+$$

$$+\int_x^{\infty}\frac{n^2}{T^{3/2}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(\underline{v}-\underline{u})^2\right\}\Bigg](\underline{v}-\underline{u})^2\,d\underline{v}dx. \quad (13)$$

Integration of the equations (11), (12) and (13) with respect to v_y and v_z yields

$$n(x) = A\sqrt{\frac{m}{2k\pi}}\Bigg[\int_{-\infty}^{x}\frac{n^2}{T^{\frac{1}{2}}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}+$$

$$+\int_x^{\infty}\frac{n^2}{T^{\frac{1}{2}}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}\Bigg]dv_x\,dx'. \quad (14)$$

$$n(x)u(x) = A\sqrt{\frac{m}{2k\pi}}\Bigg[\int_{-\infty}^{x}\frac{n^2}{T^{\frac{1}{2}}}\int_0^{\infty}\exp\left\{-\int_{x'}^{x'}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}+$$

$$+\int_x^{\infty}\frac{n^2}{T^{\frac{1}{2}}}\int_0^{\infty}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}\Bigg]dv_x dx'. \quad (15)$$

$$n(x)T(x) = \frac{2A}{3}\sqrt{\frac{m}{2k\pi}}\int_{-\infty}^{x}n^2T^{\frac{1}{2}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}+$$

$$+\int_x^{\infty}n^2T^{\frac{1}{2}}\int_0^{\infty}\frac{1}{v_x}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}\Bigg]dv_x\,dx' +$$

$$+\frac{2A}{3\sqrt{\pi}}\left(\frac{m}{2k}\right)^{3/2}\Bigg[\int_{-\infty}^{x}n^2T^{-\frac{1}{2}}\int_0^{\infty}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}+$$

$$+\int_x^{\infty}n^2T^{-\frac{1}{2}}\int_0^{\infty}\exp\left\{-\int_{x'}^{x}\frac{An}{v_x}dx'' - \frac{m}{2kT}(v_x-u)^2\right\}\Bigg]v_x dv_x\,dx' - \frac{mn(x)u^2}{3k}. \quad (16)$$

Equations (14)-(16) represent a set of singular, non-linear integral equations. "Conservation of mass" relation is used to express n in terms of u and to eliminate one of the integral equations. In the following section small perturbation technique for solving the system (14)-(16) is developed.

3. *General Method of Solution.*

The disturbance number density, temperature and velocity ν, τ and ξ are introduced in a manner similar to that of Takao [13], but with suitable modifications as follows (see the Appendix A):

$$\left.\begin{aligned} n &= n_o(1 - \epsilon\nu) \\ T &= T_o\left[1 + \epsilon(1-\gamma)\tau\right] \\ u &= u_o(1 + \epsilon\xi) \end{aligned}\right\} \tag{17}$$

where n_o, T_o, u_o are some reference number density, temperature and mass velocity which may be chosen as

$$\left.\begin{aligned} T_o &= \tfrac{1}{2}(T_1 + T_2) \\ n_o &= \tfrac{1}{2}(n_1 + n_2) \\ u_o &= \tfrac{1}{2}(u_1 + u_2) \end{aligned}\right\} \tag{18}$$

where n_1, n_2, u_1, u_2, T_1 and T_2 have the same meaning as shown in the Figure.

Further ϵ is very small and is written as

$$\epsilon = \frac{u_1 - u_2}{u_1 + u_2}. \tag{19}$$

and $\quad |\nu| \leqq 1, \ |\tau| \leqq 1, \ |\xi| \leqq 1.$ (20)

Again we expand ν, τ and ξ in powers of ϵ as

$$\left.\begin{aligned} \nu &= \sum_{n=0}^{\infty} \nu_n(\bar{x})\epsilon^n \\ \tau &= \sum_{n=0}^{\infty} \tau_n(\bar{x})\epsilon^n \\ \xi &= \sum_{n=0}^{\infty} \xi_n(\bar{x})\epsilon^n \end{aligned}\right\} \tag{21}$$

where n is now summed up and should not be confused with "n" denoting the density of the gas.

Further $x = x_o + \bar{x}/\epsilon$. Our method consists of expanding the equations (14), (15) and (16) in powers of ϵ and from these expansions we get the governing equations for ν_o, τ_o and ξ_o; the solution of which will be sought.

Left-hand sides of equations (14)-(16) are given respectively in terms of ϵ as follows:

$$n = n_o(1 - \epsilon\nu) \tag{22}$$

$$nu = n_o u_o\left[1 + \epsilon(\xi - \nu) - \epsilon^2\xi\nu\right] \tag{23}$$

and $$nT = n_o T_o\left[1 + \epsilon\left\{(1-\gamma)\tau - \nu\right\} - \epsilon^2(1-\gamma)\nu\tau\right] \tag{24}$$

It is fit to remark here that in the proceding analysis integral equation (14) is not used and instead we take advantage of the remark made at the end of the last section.

Expressing the right-hand sides of the equations (15) and (16) in powers of ϵ we get respectively

$$1+\epsilon\xi-\epsilon\nu-\epsilon^2\xi\nu=\frac{An_o}{u_o}\sqrt{\frac{m}{2k\pi T_o}}\left[\int_{-\infty}^{\bar{x}}\int_{0}^{\infty}\exp\left\{-\frac{An_o}{\epsilon v_x}(\bar{x}-\bar{x}')-\frac{m(v_x-u_o)^2}{2kT_o}\right\}\times\right.$$

$$\times\left[1+\epsilon L_1+\epsilon^2 L_2+\ldots\right]\frac{dx'dv_x}{\epsilon}-\int_{\bar{x}}^{\infty}\int_{0}^{\infty}\exp\left\{-\frac{An_o}{\epsilon v_x}(\bar{x}-\bar{x}')-\frac{m(v_x+u_o)^2}{2kT_o}\right\}\times$$

$$\left.\times\left\{\text{similar expression}\right\}\frac{d\bar{x}'dv_x}{\epsilon}\right], \tag{25}$$

$$1+\frac{mu_o^2}{3kT_o}+\epsilon\left[(1-\gamma)\tau+\frac{2mu_o^2}{3kT_o}\xi-\nu-\frac{mu_o^2}{3kT_o}\nu\right]+\epsilon^2\left[\frac{mu_o^2}{3kT_o}\xi^2-\frac{2mu_o^2}{3kT_o}\xi\nu-\right.$$

$$\left.-(1-\gamma)\nu\tau\right]=\frac{mAn_o}{3kT_o}\sqrt{\frac{m}{2k\pi T_o}}\left[\int_{-\infty}^{\bar{x}}\int_{0}^{\infty}\exp\left\{-\frac{An_o}{\epsilon v_x}(\bar{x}-\bar{x}')-\frac{m(v_x-u_o)^2}{2kT_o}\right\}\times\right.$$

$$\times\left[1+\epsilon L_1+\epsilon^2 L_2+\ldots\right]\frac{v_x d\bar{x}'dv_x}{\epsilon}+\int_{\bar{x}}^{\infty}\int_{0}^{\infty}\exp\left\{-\frac{An_o}{\epsilon v_x}(\bar{x}-\bar{x}')-\frac{m(v_x+u_o)^2}{2kT_o}\right\}\times$$

$$\times\left\{\text{similar terms}\right\}\frac{v_x d\bar{x}'dv_x}{\epsilon}+\frac{2An_o}{3}\sqrt{\frac{m}{2k\pi T_o}}\left[\int_{-\infty}^{\bar{x}}\int_{0}^{\infty}\exp\left\{-\frac{An_o}{\epsilon v_x}(\bar{x}-\bar{x}')-\right.\right.$$

$$\left.-\frac{m(v_x-u_o)^2}{2kT_o}\right\}\left[1+\epsilon L_3+\epsilon^2 L_4+\ldots\right]\frac{d\bar{x}'dv_x}{\epsilon v_x}+$$

$$+\int_{x}^{\infty}\int_{0}^{\infty}\exp\left\{-\frac{An_o}{\epsilon v_x}(\bar{x}-\bar{x}')-\frac{m(v_x+u_o)^2}{2kT_o}\right\}\left[\text{similar terms}\right]\frac{d\bar{x}'dv_x}{\epsilon v_x}, \tag{26}$$

where L_1, L_2, L_3 and L_4 etc. etc. are functions of the variables ξ, ν, τ and $\int_{\bar{x}'}^{\bar{x}}\nu dx''$. Moreover, the occurrence of these variables in the above-mentioned functions is of the same order as that of ϵ in that respective function.

Integrating the above equations by parts and after some tedious calculations equations (25), (26) give rise to the following equations respectively:

$$2\gamma\frac{d\xi}{dx}-(1-\gamma)\frac{d\nu}{dx}+(1-\gamma)\frac{d\tau}{dx}=\frac{u_o\epsilon}{An_o}\left[3(\gamma+1)\frac{d^2\xi}{dx^2}-(\gamma+3)\frac{d^2\nu}{dx^2}+3(1-\gamma)\frac{d^2\tau}{dx^2}\right]+$$

$$+\epsilon\left[(1-\gamma)\imath\frac{d\nu}{dx}+(1-\gamma)\tau\frac{d\nu}{dx}+2\nu\xi\left(\frac{d\nu}{dx}-\frac{d\xi}{dx}\right)\right] \tag{27}$$

$$(3\gamma+5)\frac{d\xi}{dx} - (\gamma+5)\frac{d\nu}{dx} + 5(1-\gamma)\frac{d\tau}{dx} = \frac{\epsilon u_o}{An_o}\left[4\gamma(\gamma+4)\frac{d^2\xi}{dx^2} + (\gamma^2+8\gamma+5)\frac{d^2\nu}{dx^2} + \right.$$

$$\left. + (10-2\gamma-8\gamma^2)\frac{d^2\tau}{dx^2}\right] + \epsilon\left[(\gamma+5)\nu\frac{d\nu}{dx} - 5(1-\gamma)\xi\frac{d\tau}{dx} + 5(1-\gamma)\tau\left(\frac{d\nu}{dx} - \frac{d\xi}{dx}\right) + \right.$$

$$\left. + 2\nu\xi\left(\frac{d\nu}{dx} - \frac{d\xi}{dx}\right)\right]. \qquad (28)$$

Putting the series (21) in the above two equations we have up to $0(\epsilon)$

$$2\gamma\frac{d\xi_1}{dx} - (1+\gamma)\frac{d\nu_1}{dx} + (1-\gamma)\frac{d\tau_1}{dx} = \frac{u_o}{An_o}\left(\gamma-\frac{1}{3}\right)\frac{d^2\xi_o}{dx^2} + \frac{d\xi_o^2}{dx} \qquad (29)$$

$$(3\gamma+5)\frac{d\xi_1}{dx} - (\gamma+5)\frac{d\nu_1}{dx} + 5(1-\gamma)\frac{d\tau_1}{dx} = \frac{u_o}{An_o}(\gamma-1)\frac{d^2\xi_o}{dx^2} + 5\frac{d\xi_o^2}{dx} \qquad (30)$$

with $$(5-3\gamma)\frac{d\xi_o}{dx} = 0 \qquad (31)$$

From equation (31) we have $\gamma = 5/3$ for $\frac{d\xi_o}{dx} \neq 0$, the value which comes out of the analysis - confirming the physical fact underlying the B-G-K model itself. Moreover, in order to satisfy the Rankine-Hugoniot relations we must have $\nu_o = \tau_o = \xi_o$ and this fact has been exploited in the above equations (29) and (30). Taking into account the remark made at the end of the last section, i. e. one of the integral equations can be eliminated with the aid of conservation of mass relation, we finally arrive at the following governing equations:

$$\frac{10}{3}\frac{d\xi_1}{dx} - \frac{8}{3}\frac{d\nu_1}{dx} - \frac{2}{3}\frac{d\tau_1}{dx} = \frac{u_o}{An_o}\cdot\frac{4}{3}\frac{d^2\xi_o}{dx^2} + \frac{d\xi_o^2}{dx} \qquad (32)$$

$$10\frac{d\xi_1}{dx} - \frac{20}{3}\frac{d\nu_1}{dx} - \frac{10}{3}\frac{d\tau_1}{dx} = \frac{u_o}{An_o}\cdot\frac{2}{3}\frac{d^2\xi_o}{dx^2} + 5\frac{d\xi_o^2}{dx} \qquad (33)$$

$$\xi_1 - \nu_1 = \xi_o^2 \qquad (34)$$

for $\gamma = 5/3$.

With the help of (34) equations (32) and (33) reduce to

$$\frac{10}{3}\frac{d\xi_o^2}{dx} + \frac{2}{3}\left(\frac{d\nu_1}{dx} - \frac{d\tau_1}{dx}\right) = \frac{u_o}{An_o}\cdot\frac{4}{3}\frac{d^2\xi_o}{dx^2} + \frac{d\xi_o^2}{dx} \qquad (35)$$

$$10\frac{d\xi_o^2}{dx} + \frac{10}{3}\left(\frac{d\nu_1}{dx} - \frac{d\tau_1}{dx}\right) = \frac{u_o}{An_o}\cdot\frac{2}{3}\frac{d^2\xi_o}{dx^2} + 5\frac{d\xi_o^2}{dx} \qquad (36)$$

From equations (35) and (36) it follows that

$$\frac{20}{3}\frac{d\xi_o^2}{dx} = \frac{u_o}{An_o}\cdot 6\,\frac{d^2\xi_o}{dx^2} \tag{37}$$

or
$$\frac{d^2\xi_o}{dx^2} = \frac{10}{9}\,\frac{An_o}{u_o}\,\frac{d\xi_o^2}{dx} \tag{38}$$

After integration equation (38) results as

$$\xi_o = \frac{1 - c\exp\left\{\dfrac{20\,An_o x}{9u_o}\right\}}{1 + c\exp\left\{\dfrac{20\,An_o x}{9u_o}\right\}} \tag{39}$$

where c is an arbitrary constant. This completes the solution of the problem.

Finally it is concluded that the method of small perturbation technique is suitable for studying the structure of weak shock-waves and the results seem to agree with the N-S solutions.

APPENDIX A

Expansion of n, T and u in terms of $\epsilon = \dfrac{u_1 - u_2}{u_1 + u_2}$.

We write n as

$$n = \frac{n_1 + n_2}{2} + \frac{n_1 - n_2}{2}\,\nu$$

$$= \frac{n_1 + n_2}{2}\left(1 + \frac{n_1 - n_2}{n_1 + n_2}\,\nu\right)$$

$$= n_o\left(1 + \frac{u_2 - u_1}{u_1 + u_2}\,\nu\right) \quad \text{for} \quad n_1u_1 = n_2u_2.$$

Hence

$$n = n_o\,(1 - \epsilon\nu) \quad \text{where} \quad \epsilon = \frac{u_1 - u_2}{u_1 + u_2}. \tag{I}$$

Similarly

$$T = \frac{T_1 + T_2}{2} + \frac{T_1 - T_2}{2}\,\tau$$

$$= T\;\left(1 + \frac{T_1 - T_2}{T_1 + T_2}\,\tau\right) \tag{II}$$

We have to express $\frac{T_1 - T_2}{T_1 + T_2}$ in powers of ϵ and in order to do that we use the normal shock relations. We know that

$$\frac{u_1 - u_2}{u_1 + u_2} = \epsilon$$

or $$\frac{u_2}{u_1} = \frac{1 - \epsilon}{1 + \epsilon} . \tag{III}$$

Further the normal shock relations are

$$\frac{u_2}{u_1} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)M_1^2} \tag{IV}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \tag{V}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} . \frac{u_2}{u_1} . \tag{VI}$$

From (IV) we obtain

$$M_1^2 = \frac{1 + \epsilon}{1 - \epsilon\gamma}$$

The substitution of which in (V) yields

$$\frac{p_2}{p_1} = \frac{1 + \gamma\epsilon}{1 - \epsilon\gamma} .$$

Consequently (VI) gives

$$\frac{T_2}{T_1} = \frac{(1 + \epsilon\gamma)(1 - \epsilon)}{(1 - \epsilon\gamma)(1 + \epsilon)}$$

or $$\frac{T_1 - T_2}{T_1 + T_2} = \frac{(1 + \epsilon\gamma)(1 + \epsilon) - (1 + \epsilon\gamma)(1 - \epsilon)}{(1 - \epsilon\gamma)(1 + \epsilon) - (1 + \epsilon\gamma)(1 - \epsilon)}$$

$$= \frac{\epsilon(1 - \gamma)}{1 - \epsilon^2\gamma} .$$

$$\cong \epsilon(1 - \gamma) \text{ neglecting higher powers of } \epsilon. \tag{VII}$$

Thus substituting the value of $(T_1 - T_2)/(T_1 + T_2)$ in the expression for T we obtain

$$T = T_o\left[1 + \epsilon(1 - \gamma)\tau\right] . \tag{VIII}$$

Carrying out the similar procedure we have

$$u = u_o(1 + \epsilon\xi). \tag{IX}$$

APPENDIX B

Integrals used in the simplification of Equations (25) and (26)

It seems sufficient to mention the type of integrals which we encounter in our analysis. Firstly, we come across the following type of integral viz.

$$\int_{-\infty}^{\bar{x}} \exp\left\{\frac{-An_o}{\epsilon v_x}(\bar{x}-\bar{x}')\right\}\left[L(\bar{x}',\epsilon)\right]d\bar{x}', \qquad \text{(i)}$$

where L is some function of $\bar{x}'$ and ϵ. Integrating (i) by parts one gets

$$L(\bar{x},\epsilon)\frac{\epsilon v_x}{An_o} - \frac{\epsilon v}{An_o}\int_{-\infty}^{\bar{x}} \exp\left\{\frac{-An_o}{\epsilon v_x}(\bar{x}-\bar{x}')\right\}\frac{dL(\bar{x}',\epsilon)}{d\bar{x}'}d\bar{x}'$$

continuing this process one obtains

$$L(\bar{x},\epsilon)\frac{\epsilon v_x}{An_o} - \left(\frac{\epsilon v_x}{An_o}\right)^2\frac{dL}{d\bar{x}} + \left(\frac{\epsilon v_x}{An_o}\right)^2\int_{-\infty}^{\bar{x}} \exp\left\{\frac{-An_o}{\epsilon v_x}(\bar{x}-\bar{x}')\right\}\frac{d^2L}{d\bar{x}'^2}d\bar{x}'.$$

It is to be noted, however, that partial integration is repeated so often that the remaining integral is of sufficient small order in ϵ so as to be neglegible.

Secondly, we have the integral

$$\int_0^\infty x^n e^{-\lambda x^2}dx \qquad \text{(ii)}$$

where n is an integer. This integral is well-known to the people working in kinetic theory of gases.

Further the expressions for L_1 and L_2 are as follows:

$$L_1 = \frac{mu_o^2}{kT_o}(v_x-u_o)\xi + \frac{m(1-\gamma)}{2kT_o}(v_x-u_o)^2\tau + \frac{n_o}{\epsilon v_x}\int_{x'}^{x}\nu dx'',$$

$$L_2 = \frac{n_o^2}{\epsilon^2 v_x^2}\left[\int_{x'}^{x}\nu dx''\right]^2 + \frac{mn_o u_o}{kT_o\epsilon v_x}(v_x-u_o)\xi\int_{x'}^{x}\nu dx'' +$$
$$+ \frac{mn_o(1-\gamma)}{2kT_o\epsilon v_x}(v_x-u_o)^2\tau\int_{x'}^{x}\nu dx'' + \frac{m^2u_o^2}{2k^2T_o^2}(v_x-u_o)^2\xi^2 -$$
$$- \frac{mu_o^2}{2kT_o}\xi^2 + \frac{mu_o^2(1-\gamma)}{2k^2T_o^2}(v_x-u_o)^2\xi\tau - \frac{mu_o(1-\gamma)}{kT_o}(v_x-u_o)\xi\tau +$$
$$+ \frac{m^2(1-\gamma)^2}{2k^2T_o^2}(v_x-u_o)^4\tau^2 - \frac{m(1-\gamma)^2}{2kT_o}(v_x-u_o)^2\tau^2.$$

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References.

1. Hayes, W.D., Gas Dynamic Discontinuities, Princeton University Press, Princeton, New Jersey, 1960.
2. Lighthill, M.J., Surveys in Mechanics, edited by G.K.Batchelor and R.M.Davies, Cambridge University Press, New York, 1956.
3. Sherman, F.S. and Talbot, L., Rarefied Gas Dynamics, edited by F.N.Devienne, Pergamon Press, New York, 1960.
4. Glansdorff, P., (1961), In Rarefied Gas Dynamics (Talbot, L. ed.), Academic Press, New York.
5. Chahine, T. Moustafa (1963), In Rarefied Gas Dynamics (J.A.Lauermann, ed.), Academic Press, New York.
6. Haviland, J.K. (1963), In Rarefied Gas Dynamics (J.A.Lauermann, ed.), Academic Press, New York.
7. Chahine, M.T. and Narasimha, R., (1965), In Rarefied Gas Dynamics (J.H. de Leeuw, ed.), Academic Press, New York.
8. Oguchi, P., (1965), In Rarefied Gas Dynamics (J.H. de Leeuw, ed.), Academic Press, New York.
9. Holway, H. Lowell, (1965), In Rarefied Gas Dynamics (J.H. de Leeuw, ed.), Academic Press, New York.
10. Sarin, S.L., General Report on the Methods of Solution of the Boltzmann Equation, Rapp. Math. Inst. T.H.Delft, May, 1965.
11. Bhatnagar, P.L., Gross, E.P. and Krook, M., A model for Collision Process in Gases. I. small amplitude process in charged and neutral one-compouent systems, Phys.Rev., Vol. 94, No. 3, P. 511 (1954).
12. Liepmann, H.W., Narasimha, R. and Chahine, M.T., Structure of plane shock layer, Phys.Fluids, vol. 5, No. 11, P. 1313 (1962).
13. Takao, A., Kinetic Theory of Plane Couette Flow of Rarefied Gases, Trans. Jap. Soc. Aero and Sp.Sc. Vol. 5, No. 7, 1962.

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